EG2008 – Series

Fourier Series

Fourier Series (pg 231)

- Named after the Mathematician, J.B.J Fourier of France in 1812
- The Fourier Series for a periodic function $f(t)$ with period T , and circular frequency $\omega = 2\pi/T$, is given by

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega t + \sum_{n=1}^{\infty} b_n \sin n\omega t$$

where the Fourier coefficients are given as

$$a_n = \frac{2}{T}\int_{d}^{d+T} f(t)\cos n\omega t dt \quad (n = 0,1,2,\ldots)$$

$$b_n = \frac{2}{T}\int_{d}^{d+T} f(t)\sin n\omega t dt \quad (n = 1,2,\ldots)$$

Benefits of knowing whether *f*(*t*) is odd or even functions (pg 239)

- If $f(t)$ is an even function, then the Fourier series consists of the constant term and the cosine terms only

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega t \qquad b_n = 0$$

- where

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos n\omega t\,dt = \frac{4}{T}\int_{0}^{T/2} f(t)\cos n\omega t\,dt \quad \text{for} \quad n = 0,1,2,3\ldots$$

Benefits of knowing whether *f*(*t*) is odd or even functions

- If $f(t)$ is an odd function, then the Fourier series consists of the sine terms only

$$f(t) = \sum_{n=1}^{\infty} b_n \sin n\omega t$$

- where $a_0 = 0$ and $a_n = 0$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin n\omega t\,dt = \frac{4}{T}\int_{0}^{T/2} f(t)\sin n\omega t\,dt$$

Convergence of Fourier Series

T/F Question:

If we can find a_n and b_n, then the Fourier Series obtained converges to the given periodic function.

Answer:

F

Convergence of Fourier Series (pg 245)

- Dirichlet's conditions ensures f(t) has a convergent Fourier Series expansion

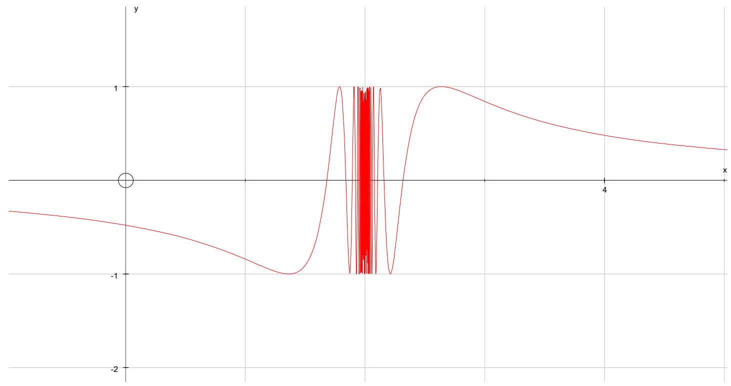
If f(t) is a bounded periodic function that in any period has

(a) a finite number of isolated maxima and minima, and
(b) a finite number of points of finite discontinuity

then the Fourier series expansion of f(t) converges to f(t) at all points

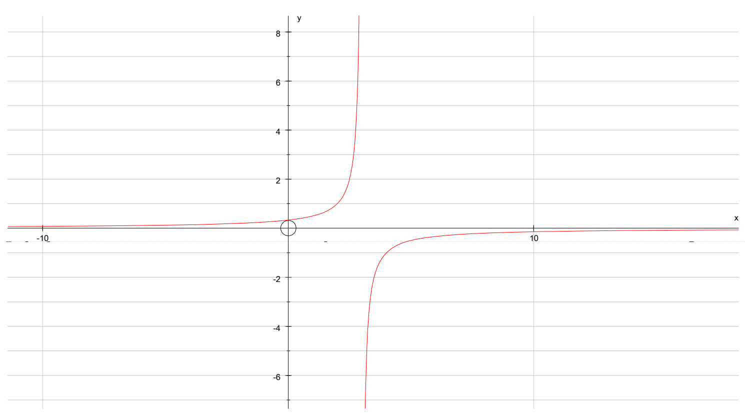
Example 4.8

(a) $\sin\left(\dfrac{1}{t-2}\right)$ for $0 < t < 2\pi$



Example 4.8

(b) $\dfrac{1}{3-t}$ for $0 < t < 2\pi$



Rate of Convergence for Fourier Series

Question:

How many terms must be taken in the Fourier Series expansion in order to obtain a realistic approximation to the periodic function f(t)?

Rate of Convergence for Fourier Series

- If f(t) is only piecewise-continuous, then the coefficients in its Fourier series representation decrease as 1/n.
- That is, it may be necessary to include a large number of terms to obtain an adequate approximation to f(t).

$$f(t) = \pi - 2(\sin t + \frac{\sin 2t}{2} + \frac{\sin 3t}{3} + ...)$$

Rate of Convergence for Fourier Series

- If f(t) is a continuous function but has discontinuous first derivatives (due to sharp corners), then the coefficients in its Fourier series representation decrease as $1/n^2$.
- That is, the series converges more rapidly and requires lesser number of terms.

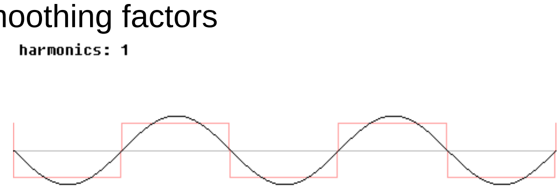
$$f(t) = \frac{5\pi}{16} - \frac{2}{\pi}\left(\cos t + \frac{\cos 3t}{t^2} + \dots\right)$$

$$-\frac{2}{\pi}\left(\frac{\cos 2t}{2^2} + \frac{\cos 6t}{6^2} + \dots\right)$$

$$+\frac{1}{\pi}\left(\sin t - \frac{\sin 3t}{3^2} + \dots\right)$$

Gibb's Phenomenon (pg 248)

- Occurrence of an undershoot and an overshoot at points of discontinuity of f(t)
- The magnitude of the undershoot/overshoot does not diminish as $n \to \infty$, but simply get 'sharper' and 'sharper', tending to a peak
- The spikes need to be suppressed using smoothing factors



Discrete Frequency Spectra (pg 280)

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{2n\pi t}{T}+b_n\sin\frac{2n\pi t}{T}\right)$$

$$=\frac{a_0}{2}$$

$$+\left(a_1\cos\frac{2\pi t}{T}+b_1\sin\frac{2\pi t}{T}\right)$$ First harmonic (f)

$$+\left(a_2\cos\frac{4\pi t}{T}+b_2\sin\frac{4\pi t}{T}\right)$$ Second harmonic (2xf)

$$+\left(a_3\cos\frac{6\pi t}{T}+b_3\sin\frac{6\pi t}{T}\right)$$ Third harmonic (3xf)

$$+\ldots$$

$$+\left(a_n\cos\frac{2n\pi t}{T}+b_n\sin\frac{2n\pi t}{T}\right)$$ n^{th} harmonic (nxf)

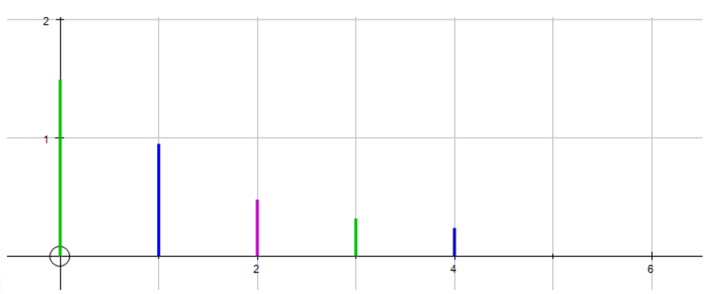
$$+\ldots.$$

Annex AA

- The Fourier Series expansion for a triangular wave is given as

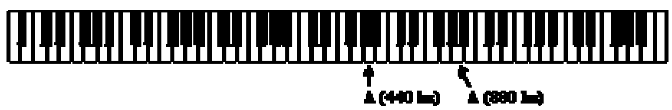
$$f(t)=1.5+0.95\sin\omega t+0.48\sin 2\omega t$$

$$+0.32\sin 3\omega t+0.24\sin 4\omega t+\ldots$$



Engineering Term in Fourier Series

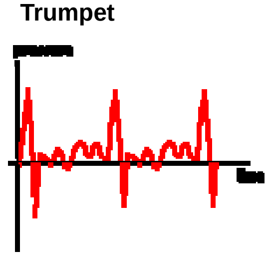
- The Fourier series can tell us the strength of frequencies in a periodic function, eg. sound from a musical instrument:
  - The **loudness** of the note is measured by the *magnitude* of the changes in air pressure (a_1, b_1, a_2, b_2, … a_n, b_n,…).
  - The **pitch** of the note is the frequency (cycles/second) of repetition of the basic pressure pattern. Human hearing is confined to frequencies that range roughly from 20 to 18,000 hertz (1/period)



Engineering Term in Fourier Series

- The Fourier series can tell us the strength of frequencies in a periodic function, eg. sound from a musical instrument:
  - The **timbre** of the note includes those characteristics that enable us to tell a piano note from a violin note with the same loudness and pitch.

Trumpet



Tuning Fork